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THE REMAINDER IN WEYL'S LAW FOR HEISENBERG MANIFOLDS

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Abstract

We prove that the error term $R(\lambda)$ in Weyl's law is $\mathcal{O}_\epsilon(\lambda^{5/6+\epsilon})$ for certain three-dimensional Heisenberg manifolds. We also show that the L^2-norm of the Weyl error term integrated over the moduli space of left-invariant Heisenberg metrics is $\ll \lambda^{3/4+\epsilon}$. We conjecture that $R(\lambda) = \mathcal{O}_\epsilon(\lambda^{3/4+\epsilon})$ is a sharp deterministic upper bound for Heisenberg three-manifolds.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension n with Laplace-Beltrami operator Δ and spectral counting function

$$N(\lambda) := \#\{\lambda_j \in \mathrm{Spec}(\Delta);\ \lambda_j \leq \lambda\}.$$

Then, a celebrated theorem of Hörmander [16] asserts that

$$N(\lambda) = c_n \mathrm{vol}(M)\, \lambda^{n/2} + \mathcal{O}(\lambda^{(n-1)/2}), \tag{1.1}$$

for some constant c_n depending only on the dimension. Moreover, the estimate in (1.1) is sharp as can be seen by considering the round sphere, S^n. The question of determining the optimal bound for the error term

$$R(\lambda) := N(\lambda) - c_n \mathrm{vol}(M) \lambda^{n/2}$$

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in any given example is a difficult problem which depends on the properties of the associated geodesic flow, and is far from being understood in detail.

Duistermaat and Guillemin improved the estimate on the error term for a (generic) manifold in [8]. This result states that if the (Liouville) measure of the set of unit-speed geodesics in S^*M is zero, then

$$R(\lambda) = o(\lambda^{(n-1)/2}).$$

Perhaps the simplest class of examples satisfying the above conditions are the completely integrable geodesic flows. Under mild non-degeneracy assumptions, the existence of action-angle variables implies that on a dense open subset of T^*M the phase-space is foliated by Lagrangian tori. For tori with rational slope, the geodesics are all periodic, whereas for the irrational ones they are dense in the corresponding torus. Since the conditions of [8] are therefore satisfied, it is natural to ask for explicit improvements to the $o(\lambda^{n-1/2})$ error term for integrable geodesic flows. There are very few explicit results regarding the error term $R(\lambda)$. The simplest example of an integrable geodesic flow on a surface is that of a flat metric on a torus, $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Here, there is a famous conjecture of Hardy which asserts that for any $\epsilon > 0$,

$$R(\lambda) = \mathcal{O}_\epsilon(\lambda^{1/4+\epsilon}). \tag{1.2}$$

There is much evidence to suggest that the bound in (1.2) is optimal. Cramér [4] proved that averaging over the spectrum is consistent with (1.2):

$$\lim_{T\to\infty} \frac{1}{T^{3/2}} \int_0^T |R(\lambda)|^2 \, d\lambda = c > 0.$$

Hardy [14] proved that $R(\lambda) = \Omega(\lambda^{1/4})$. In [18], we showed that the variance of $R(\lambda)$ integrated over the moduli space of flat tori is consistent with the estimate (1.2). The same result has been recently reproved by Hofmann and Iosevich with more elementary techniques [15].

Another case with completely integrable geodesic flow where improvements on Weyl's law are known is the case of (generic) convex surfaces of revolution. Colin de Verdière [5] showed that

$$R(\lambda) = \mathcal{O}(\lambda^{1/3}).$$

He also explained that the spectral counting problem in that case is reduced to a certain lattice-point counting problem.

The purpose of this article is to study Weyl's law for 3-dimensional Heisenberg manifolds. Recently, Butler [2, Sect. 5] showed that Heisenberg (and more general nilmanifolds) are completely integrable in the sense of Liouville and nondegenerate. Moreover, one integral is C^∞ but *not* real-analytic, while the rest are algebraic. In fact, using earlier work of Taimanov [20], Butler shows that because of topological constraints on $\pi_1(M)$, one *cannot* construct analytic integrals for such manifolds. In related work, Bolsinov and Taimanov [1] have constructed integrable geodesic flows similar to the Heisenberg examples with the property that on certain Lagrangian invariant tori the geodesic flow is a cat map. Remarkably, these latter examples are shown to have positive topological entropy. Heisenberg manifolds and, more generally, nilmanifolds have been extensively studied in the context of isospectrality, because of their rich structure, see [12], [11], [6], [7], [10].

In light of the integrability of geodesic flow on Heisenberg manifolds and the result in (1.1) it is natural to ask for explicit improvements in the Weyl error for such manifolds. To the best of our knowledge no results or conjectures have been made regarding the remainder in Weyl's law for Heisenberg manifolds.

We now introduce notation and state our results. The 3-dimensional Heisenberg group H_1 consists of all matrices of the form

$$\gamma(x,y,t) = \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x,y,t \in \mathbb{R}.$$

We are interested in the spectrum of Heisenberg manifolds. These are defined as $(\Gamma\backslash H_1, g)$, where Γ is a discrete subgroup of H_1 with compact quotient and where g is a left H_1-invariant metric. The classification theorem in [12, 2.4] allows us to restrict out attention to subgroups Γ_r of the following type

$$\Gamma_r = \{\gamma(x,y,t) : x \in r\mathbb{Z}, y \in \mathbb{Z}, t \in \mathbb{Z}\}.$$

The left invariant metrics on H_1 are determined by the induced inner product on the Lie algebra $\mathcal{H}_1$. We can replace the metric g with $\phi^* g$, where ϕ is an inner automorphism, in such a way that the direct sum split of the Lie algebra $\mathcal{H}_1 = \mathbb{R}^2 + \mathfrak{z}$ is orthogonal, see [12, 2.6(b)]. Here $\mathfrak{z}$ is the center of the Lie algebra and

$$\mathbb{R}^2 \equiv \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x,y \in \mathbb{R} \right\}.$$

With respect to this orthogonal split of $\mathcal{H}_1$ the metric g has the form

$$g = \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{12} & h_{22} & 0 \\ 0 & 0 & g_3 \end{pmatrix}$$

where $g_3 > 0$ and $h_{11}h_{22} - h_{12}^2 > 0$. For simplicity we only consider the group $\Gamma = \Gamma_1$. We also restrict to the case $h_{12} = 0$ and choose parameters

$$u_i = h_{ii}^{-1}, \quad i = 1, 2, \quad u_3 = g_3^{-1}. \tag{1.3}$$

We call this metric $g(\vec{u})$. This is a subvariety of the moduli space of metrics but our methods can easily be extended to the whole moduli space.

The spectrum of the Laplace operator associated with this metric consists of two parts, see [12, p. 258]:

(1) Type I eigenvalues: these are eigenvalues of a torus T^2 with metric given by the matrix (h_{ij}), $i, j = 1, 2$. We denote this part of the spectrum by $\Sigma_1(u_1, u_2) := \{4\pi^2(u_1 m^2 + u_2 n^2);\ m, n \in \mathbb{Z}\}$,

(2) Type II eigenvalues: $\Sigma_2(u_1, u_2, u_3) := \{\mu(c, k) = 4\pi^2 u_3 c^2 + 2\pi u_1 u_2 c(2k+1);\ c, k \in \mathbb{Z}, c > 0, k \geq 0\}$. These eigenvalues have to be counted with multiplicity as follows: for every $c > 0$, the $\mu(c, k)$ is counted with multiplicity $2c$ and, if it happens that we get the same eigenvalue from different pairs (c, k), the multiplicities are added.

It is the Type II eigenvalues that contribute the main term in Weyl's law. In the deterministic result below both Type I and Type II eigenvalues contribute to the improvement of Weyl's law.

We start with an improvement of Hörmander's bound for an 'arithmetic' Heisenberg manifold, where

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2\pi \end{pmatrix}.$$

Theorem 1.1. *The Heisenberg manifold $M = (\Gamma_1 \setminus H_1, g_1)$ has Weyl Law*

$$N(\lambda) = c_3 \mathrm{vol}(M) \lambda^{3/2} + \mathcal{O}(\lambda^{5/6+\epsilon}).$$

Here $c_3 = (6\pi^2)^{-1}$ and $\mathrm{vol}(M) = \sqrt{2\pi}$, see [12, Prop. 2.9]. The choice of $g_3 = 2\pi$ was made so that we can factor 2π in $\mu(c,k)$. There is nothing particular about this choice, we could have used 4π, 6π etc. and deal with other indefinite rational quadratic forms. We prove Theorem 1.1 using Van der Corput's method. The result in Theorem 1.1 is not optimal; see our forthcoming paper [3] for improvements and the generalization to an arbitrary metric g. As we explain in Section 2, we reduce the calculation to a lattice-point counting with weight for points under the hyperbolas $xy = \lambda$, and below the line $x = y$. This is related to the divisor problem

$$\sum_{n \le x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x).$$

Here $\tau(n)$ is the number of divisors of n. Using Van der Corput's method we get the estimate $\Delta(x) = \mathcal{O}(x^{1/3+\epsilon})$, see [9, p. 69]. Improvements to this are known but they are far from the conjectured bound, $\Delta(x) = \mathcal{O}_\epsilon(x^{1/4+\epsilon})$. This would follow from certain estimates on exponential sums, see Remark 2.3, Section 2. The latter bound for these sums, implies the bound $R(\lambda) = \mathcal{O}_\epsilon(\lambda^{3/4+\epsilon})$ for the Weyl law on M, see Section 2. We are thus lead to the following conjecture:

Conjecture 1.2. The pointwise estimate $R(\lambda) = \mathcal{O}_\delta(\lambda^{3/4+\delta})$ is sharp for 3-dimensional Heisenberg manifolds.

We provide evidence for this conjecture in the following theorems. Our second theorem gives a probabilistic estimate for the error $R(\lambda)$ in the Weyl law, which is consistent with Conjecture 1.2.

Theorem 1.3. *Fix $\epsilon \in (0,1)$ and let $\vec{u} := (u_1, u_2, u_3) \in I^3$ where $I := [1-\epsilon, 1+\epsilon]$. Then, for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that for $\lambda \ge \lambda_0(\delta)$,*

$$\int_{I^3} \left| N(\lambda; \vec{u}) - \frac{1}{6\pi^2} \mathrm{vol}(M(\vec{u}))\lambda^{3/2} \right|^2 d\vec{u} \le C_\delta \, \lambda^{3/2+\delta}.$$

We take up the proof of Theorem 1.3 in Section 3. In Section 4, we review the geometry of the geodesic flow on Heisenberg manifolds and, in particular, we show that there exist 4-dimensional manifolds of periodic orbits in S^*M. In Section 5, we use the existence of these periodic manifolds together with a trace-formula argument, see [19], to get the following lower bound for the spectral average of the Weyl error function:

Theorem 1.4. *Fix $\vec{u} = (1,1,1)$ and let $\lambda \geq \lambda_0 > 0$ be sufficiently large. Then it follows that*

$$\frac{1}{\lambda}\int_{\lambda}^{2\lambda} \left| N(\tau) - \frac{1}{6\pi^2}\mathrm{vol}(M)\tau^{3/2} \right| d\tau \gg \lambda^{3/4}.$$

Thus, Theorem 1.4 yields a lower average spectral bound that is consistent with the bound in Conjecture 1.2.

Remark 1.5. Regarding Theorem 1.3, it is natural to ask whether there is anything special about using the Lebesgue measure on the parameter space. In fact, it turns out that we can use Gaussian measures concentrating around a fixed point in the moduli space to the order $\lambda^{-1/4}$. This point will be addressed in future work together with the issue of universality, i.e., whether one could get the same type of estimates for every (absolutely continuous) measure on the moduli space.

Remark 1.6. Ultimately one would like to prove Conjecture 1.2 for almost all left-invariant metrics on $\Gamma \setminus H_1$.

Remark 1.7. The situation for higher dimensional Heisenberg manifolds is not the same. It seems that the estimates on $R(\lambda)$ depend on the diophantine properties of certain numbers depending on the metric g. This is work in progress in the thesis of Khosravi.

2. Deterministic estimates: Proof of Theorem 1.1

With the choice of metric for Theorem 1.1 we see that the Type II eigenvalues are $2\pi c(c+2l+1)$, where $c > 0$ and $l \geq 0$. We rewrite these numbers as $2\pi ck$, where $k > c$ and c and k do not have the same parity. Every eigenvalue has multiplicity $2c$. To estimate $N_{\mathrm{II}}(\lambda)$ the counting function for Type II eigenvalues we need to count the pairs (c,k) with $k > c$ and with weight $2c$ and subtract the count of the pairs (c,k) with $k > c$, $k \equiv c (\mathrm{mod}\ 2)$, again with weight $2c$.

Set $\psi(u) = u - [u] - 1/2$. We set

$$N(t) = \sum_{ck \leq t, c<k} c.$$

In Figure 1 the lattice points in $\mathbb{Z}^2$ that contribute to $N(16)$ are marked by the dots and $x = k$, $y = c$. It is important that we compute two-term asymptotic expansions, since we must see the cancellation of

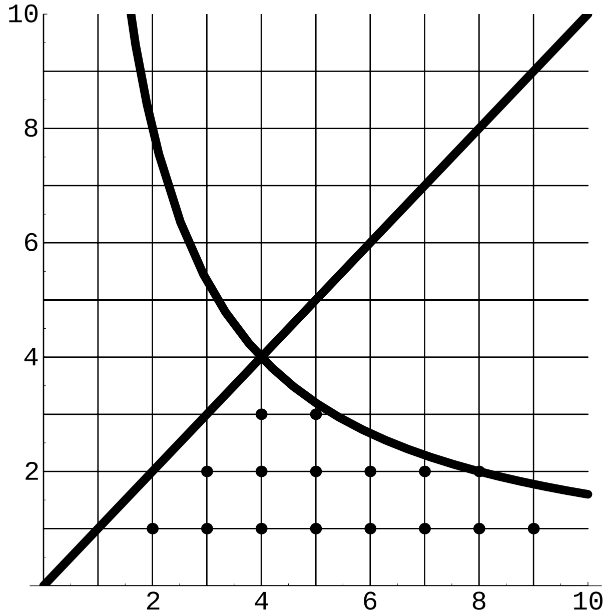

Figure 1: Lattice points under $xy = 16$ and $y = x$.

the λ^1 term that comes out of the counting function for Type I eigenvalues. We split the summation into two parts: the triangular region $0 \leq c < k \leq \sqrt{t}$ and the horn-shaped region under the hyperbola $ck = t$ and to the right of the triangle. This gives:

$$N(t) = \sum_{c<k\leq\sqrt{t}} c + \sum_{ck\leq t, k>\sqrt{t}} c = A + B. \tag{2.1}$$

It follows from the Euler summation formula [9, Satz 3, p. 187] that

$$\sum_{n\leq u} n^a = \frac{u^{a+1}}{a+1} - \psi(u)u^a + \mathcal{O}(u^{a-1}). \tag{2.2}$$

Using (2.2), we easily get that

$$\begin{aligned}
A &= \sum_{c\le\sqrt{t}} c\left([\sqrt{t}]-c\right) = [\sqrt{t}]\sum_{c\le\sqrt{t}} c - \sum_{c\le\sqrt{t}} c^2 \\
&= \left(\sqrt{t}-1/2-\psi(\sqrt{t})\right)\left(t/2-\psi(\sqrt{t})\sqrt{t}+\mathcal{O}(1)\right) \\
&\quad -\frac{t^{3/2}}{3}+\psi(\sqrt{t})t+\mathcal{O}(\sqrt{t}) \\
&= \frac{t^{3/2}}{6}+\left(-1/4-\psi(\sqrt{t})/2\right)t+\mathcal{O}(\sqrt{t}).
\end{aligned}$$

On the other hand we see that

$$\begin{aligned}
B &= \sum_{c\le\sqrt{t}} c \sum_{\sqrt{t}<k\le t/c} 1 = \sum_{c\le\sqrt{t}} c\left(t/c-\psi(t/c)-\sqrt{t}+\psi(\sqrt{t})\right) \\
&= t[\sqrt{t}]+\left(\psi(\sqrt{t})-\sqrt{t}\right)\sum_{c\le\sqrt{t}} c - \sum_{c\le\sqrt{t}} c\psi(t/c) \\
&= \frac{t^{3/2}}{2}-\frac{t}{2}+\frac{t\psi(\sqrt{t})}{2}-\sum_{c\le\sqrt{t}} c\psi(t/c)+\mathcal{O}(\sqrt{t}),
\end{aligned} \tag{2.3}$$

using the summation formula (2.2). We call the sum on (2.3) by $E(t)$ and we get

$$N(t) = A+B = \frac{2}{3}t^{3/2}-\frac{3}{4}t+E(t)+\mathcal{O}(t^{1/2}). \tag{2.4}$$

We now do the calculation on the lattice $L=\{(x,y)\in\mathbb{Z}^2, x\equiv y \bmod 2\}$. We set

$$N_L(t) = \sum_{xy\le t,(x,y)\in L, y<x} y.$$

In Figure 2 the lattice points in L that contribute to $N_L(16)$ are shown. As before we split the region into a triangle $y<x\le\sqrt{t}$ and into the horn-shaped region to the right of the triangle. For technical reasons we have to compute separately the contributions of the even x and y and the odd x and y. Calling the four contributions A_e, A_o, B_e

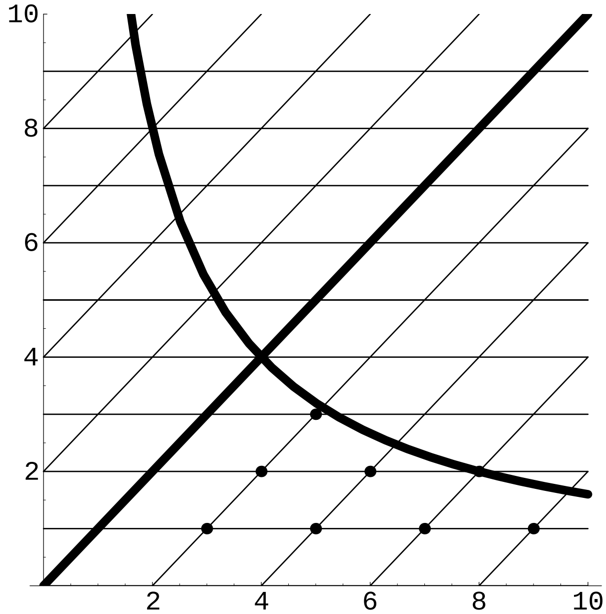

Figure 2: Lattice points (x, y), $x \equiv y \pmod 2$.

and B_o, we have $N_L(t) = A_e + A_o + B_e + B_o$. We compute:

$$\begin{aligned} A_e &= \sum_{y \le \sqrt{t}, y \in 2\mathbf{Z}} y \sum_{x \in 2\mathbf{Z}, y < x \le \sqrt{t}} 1 = 2 \sum_{y \le \sqrt{t}/2} y \sum_{y < x \le \sqrt{t}/2} 1 \\ &= 2 \sum_{y \le \sqrt{t}/2} y \left([\sqrt{t}/2] - y \right) \\ &= 2[\sqrt{t}/2] \left(\frac{t}{8} - \psi(\sqrt{t}/2) \frac{\sqrt{t}}{2} + \mathcal{O}(1) \right) \\ &\quad - 2 \left(\frac{t^{3/2}}{24} - \psi(\sqrt{t}/2) \frac{t}{4} + \mathcal{O}(\sqrt{t}) \right) \\ &= \frac{t^{3/2}}{24} + \left(-\frac{1}{8} - \frac{1}{4} \psi \left(\frac{\sqrt{t}}{2} \right) \right) t + \mathcal{O}(\sqrt{t}). \end{aligned}$$

For A_o we set $y = 2y' - 1$ and $x = 2x' - 1$ to get:

$$\begin{aligned}A_o &= \sum_{y\le\sqrt{t}, y\notin 2\mathbf{Z}} y \sum_{x\notin 2\mathbf{Z}, y<x\le\sqrt{t}} 1 = \sum_{y\le(\sqrt{t}+1)/2} (2y-1) \sum_{y<x\le(\sqrt{t}+1)/2} 1\\ &= \sum_{y\le(\sqrt{t}+1)/2} (2y-1)\left([(\sqrt{t}+1)/2]-y\right)\\ &= -[(\sqrt{t}+1)/2]^2 + \left(2[(\sqrt{t}+1)/2]+1\right)\sum_{y\le(\sqrt{t}+1)/2} y - \sum_{y\le(\sqrt{t}+1)/2} 2y^2.\end{aligned}$$

Using (2.2) again we get

$$A_o = \frac{t^{3/2}}{24} + \left(-\frac{1}{8} - \frac{1}{4}\psi\left(\frac{\sqrt{t}+1}{2}\right)\right) t + \mathcal{O}(\sqrt{t}). \tag{2.5}$$

We now compute on the horn-shaped region $\sqrt{t} < x \le t/y$:

$$\begin{aligned}B_e &= \sum_{y\le\sqrt{t}, y\in 2\mathbf{Z}} y \sum_{x\in 2\mathbf{Z}, \sqrt{t}<x\le t/y} 1 = 2\sum_{y\le\sqrt{t}/2} y \sum_{\sqrt{t}/2<x\le t/(4y)} 1\\ &= 2\sum_{y\le\sqrt{t}/2} y\left([t/(4y)] - [\sqrt{t}/2]\right)\\ &= 2\sum_{y\le\sqrt{t}/2} y\left(\frac{t}{4y} - \psi(t/(4y)) - \frac{\sqrt{t}}{2} + \psi(\sqrt{t}/2)\right)\\ &= \frac{t}{2}\sum_{y\le\sqrt{t}/2} 1 + 2\left(\psi(\sqrt{t}/2) - \frac{\sqrt{t}}{2}\right)\sum_{y\le\sqrt{t}/2} y - 2\sum_{y\le\sqrt{t}/2} y\psi\left(\frac{t}{4y}\right).\end{aligned}$$

We set

$$E_1(t) = 2\sum_{y\le\sqrt{t}/2} y\psi(t/(4y)), \tag{2.6}$$

and use (2.2) to get

$$B_e = \frac{t^{3/2}}{8} + \left(-\frac{1}{4} + \frac{1}{4}\psi(\sqrt{t}/2)\right) t - E_1(t) + \mathcal{O}(\sqrt{t}). \tag{2.7}$$

For B_o we substitute $x = 2x' - 1$ and $y = 2y' - 1$ and get:

$$
\begin{aligned}
B_o &= \sum_{y \le \sqrt{t}, y \notin 2\mathbf{Z}} y \sum_{\sqrt{t} < x \le t/y, x \notin 2\mathbf{Z}} 1 \\
&= \sum_{y \le (\sqrt{t}+1)/2} (2y-1) \sum_{(\sqrt{t}+1)/2 < x \le t/(2(2y-1))+1/2} 1 \\
&= \sum_{y \le (\sqrt{t}+1)/2} (2y-1) \Big([t/(2(2y-1)) + 1/2] - [(\sqrt{t}+1)/2] \Big) \\
&= \frac{t}{2}[(\sqrt{t}+1)/2] + \left(\psi\left(\frac{\sqrt{t}+1}{2} \right) - \frac{\sqrt{t}}{2} \right) \sum_{y \le (\sqrt{t}+1)/2} (2y-1) - E_2(t)
\end{aligned}
$$

where we set

$$
E_2(t) = \sum_{y \le (\sqrt{t}+1)/2} (2y-1) \psi\left(\frac{t}{2(2y-1)} + 1/2 \right). \tag{2.8}
$$

Finally we get

$$
B_o = \frac{t^{3/2}}{8} + \frac{1}{4} \psi\left(\frac{\sqrt{t}+1}{2} \right) t - E_2(t) + \mathcal{O}(\sqrt{t}). \tag{2.9}
$$

Remark 2.1. The main term asymptotics for the standard lattice $\mathbb{Z}^2$ and the lattice L are consistent, i.e., the main term in $A_o + A_e$ is half the main term of A and so is the main term in $B_o + B_e$ and B. However, this is not true for the term involving t^1 in $A_o + A_e$ and A.

Using Van der Corput's method we will prove that

$$
\max(|E(t)|, |E_1(t)|, |E_2(t)|) \ll t^{5/6+\epsilon}. \tag{2.10}
$$

Then

$$
N_L(t) = A_o + A_e + B_o + B_e = \frac{t^{3/2}}{3} - \frac{t}{2} - E_1(t) - E_2(t) + \mathcal{O}(\sqrt{t}). \tag{2.11}
$$

Subtracting (2.11) from (2.4) gives

$$
\frac{t^{3/2}}{3} - \frac{t}{4} + E(t) - E_1(t) - E_2(t) + \mathcal{O}(\sqrt{t}).
$$

Now we can count the eigenvalues of Δ on M that are of Type II. Each comes with multiplicity $2c$, so we double the difference of the two results and we notice that $t = \lambda/(2\pi)$. This gives

$$N_{\mathrm{II}}(\lambda) = 2\left(\frac{\lambda}{2\pi}\right)^{3/2}\frac{1}{3} - 2\frac{1}{4}\frac{\lambda}{2\pi} + \mathcal{O}(\lambda^{5/6+\epsilon}).$$

On the other hand the eigenvalues of Type I give

$$N_{\mathrm{I}}(\lambda) = \frac{\lambda}{4\pi} + \mathcal{O}(\lambda^{1/2}),$$

since the corresponding torus has area 1. We see that the λ terms cancel and we get as improvement in the Weyl Law $R(\lambda) \ll \lambda^{5/6+\epsilon}$.

2.1 Application of Van der Corput's method

For simplicity we show only the bound $E_1(t) \ll t^{5/6}\log t$. The bound for $E(t)$ and $E_2(t)$ is proved similarly. A summation by parts gives

$$E_1(t) = \sqrt{t}\sum_{n=1}^{\sqrt{t}/2}\psi\left(\frac{t}{4n}\right) - \int_1^{\sqrt{t}/2}\sum_{n\le x}\psi\left(\frac{t}{4n}\right)dx. \tag{2.12}$$

We need to show that the sums in (2.12) are $\ll t^{1/3}\log t$ for $x \le \sqrt{t}$. The main point in Van der Corput's method can be summarized in the following proposition, see [9, Satz 1, p. 41]:

Proposition 2.2. *Let $f(u)$ be a twice-differentiable function on the interval $[a,b]$ and satisfies either $f''(u) \ge \lambda$ for all $u \in [a,b]$, or, $f''(u) \le -\lambda$ for all $u \in [a,b]$, where $0 < \lambda \le 1$. Then*

$$\sum_{a\le l\le b}\psi(f(l)) \ll |f'(b) - f'(a)|\lambda^{-2/3} + \lambda^{-1/2},$$

with the implied constants being absolute.

We set $f(u) = t/(4u)$, so that $f'(u) = -t/(4u^2)$ and $f''(u) = t/(2u^3) \ge t/(2b^3)$ for $a \le u \le b$. Proposition 2.2 gives for $t/(2b^3) \le 1$

$$\begin{aligned}\sum_{a\le m\le b}\psi\left(\frac{t}{4m}\right) &\ll \left(\frac{t}{4a^2} - \frac{t}{4b^2}\right)\left(\frac{t}{2b^3}\right)^{-2/3} + \left(\frac{t}{2b^3}\right)^{-1/2}\\ &\ll t^{1/3}(a^{-2} - b^{-2})b^2 + t^{-1/2}b^{3/2}.\end{aligned} \tag{2.13}$$

We choose L to be the largest integer with $2^{-L}x \geq t^{1/3} \geq x^{2/3}$, i.e., $L \leq [\log x/(3\log 2)]$. In particular $L \ll \log x \ll \log t$. We have, using a dyadic decomposition,

$$\begin{aligned}
(2.14)\quad & \sum_{n\leq x} \psi\left(\frac{t}{4n}\right) \\
&= \sum_{n\leq 2^{-L}x} \psi\left(\frac{t}{4n}\right) + \sum_{l=0}^{L-1} \sum_{2^{-l-1}x\leq n\leq 2^{-l}x} \psi\left(\frac{t}{4n}\right) + O(L) \\
&= O(2^{-L}x) + \sum_{l=0}^{L-1} \sum_{2^{-l-1}x\leq n\leq 2^{-l}x} \psi\left(\frac{t}{4n}\right) + O(L) \\
&\ll t^{1/3} + \sum_{l=1}^{L} \left(t^{1/3}\left(\frac{(2^{-l}x)^2}{2^{-l-1}x)^2} - 1\right) + t^{-1/2}2^{-3l/2}t^{3/4}\right) + \log t \\
&\ll t^{1/3}\log t.
\end{aligned}$$

using (2.13) to estimate the inner sum in the third line of (2.14). Also we used that

$$\lambda = \frac{t}{2b^3} = \frac{t}{2(2^{-l}x)^3} \leq \frac{t}{2(t^{1/3})^3} \leq \frac{1}{2} \leq 1.$$

Remark 2.3. The conjectural best bound is

$$\sum_{n\leq\sqrt{t}} \psi(t/n) \ll t^{1/4+\epsilon},$$

which is equivalent to the conjectured result for the divisor problem. This would follow from the following estimate on exponential sums, see [17, p. 57-59]: For $[a,b] \subset [N,2N]$, $T \geq N^2$, we have

$$\sum_{n=a}^{b} e^{T/n} \ll \left(\frac{T}{N^2}\right)^{\epsilon} N^{1/2+\epsilon}.$$

This would be an application of Conjecture 2 in [17, p. 59] stating that $(\epsilon, 1/2+\epsilon)$ is an exponent pair. The relation with the function $\psi(u)$ comes through the Fourier series $\psi(u) = -\sum_{k\neq 0}\exp(2\pi iku)/(2\pi ik)$. Equation (2.12) would imply by the same method that $E_1(t) \ll t^{3/4+\epsilon}$.

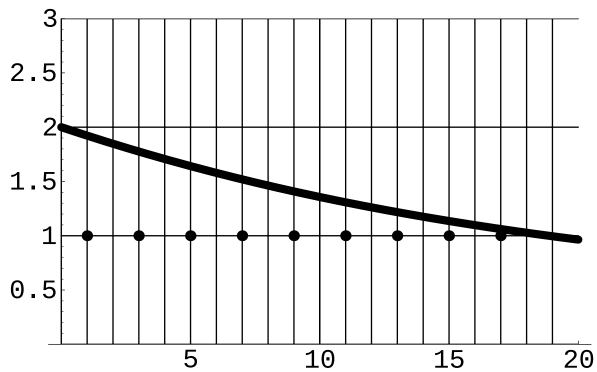

Figure 3: The lattice points contributing to Type II eigenvalues, $\lambda = 16\pi^2$.

3. Probabilistic results: Proof of Theorem 1.3.

We consider now the Heisenberg manifolds $(\Gamma_1 \setminus H_1, g(\vec{u}))$, where $g(\vec{u})$ is a perturbation of the metric given by the identity matrix. For this metric the lattice points that contribute to the spectral counting function of Type II eigenvalues $N_{\mathrm{II}}(\lambda)$ lie below the hyperbola $2\pi y(2\pi y + x) = \lambda$ and x is an odd integer, while $c = y > 0$. For $\lambda = 16\pi^2 \approx 157.9$ these points are shown in Figure 3.

3.1 Averaged density of states

In this section we give an asymptotic estimate for the averaged density of states of the eigenvalues, λ_j, $j = 1, 2, ...$ of the Laplacian Δ on a Heisenberg manifold, $M(\vec{u}) = (\Gamma \setminus H_1, g(\vec{u}))$. We put $\hbar^{-1} = \sqrt{\lambda}$. Most of our estimates will be given in terms of $\hbar$ but the reader should have no difficulty in expressing them in terms of λ. We define the average density of states as follows:

$$\mathrm{AV}(\phi) = \sum_{j=1}^{\infty} \int_{I^3} \phi(\lambda_j(\vec{u}) - \lambda)\, d\vec{u} \tag{3.1}$$
$$= \mathrm{AV}_1(\phi) + \mathrm{AV}_2(\phi),$$

where,

$$\text{(3.2)} \qquad \mathrm{AV}_1(\phi) := \hbar \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \int_{I^3} \int_{\mathbb{R}} e^{is[H(m\hbar,n\hbar;\vec{u})-1]/\hbar}\, \check{\phi}(\hbar s)\, ds\, d\vec{u},$$

$$\mathrm{AV}_2(\phi) := \hbar \sum_{c\in\mathbb{Z}^+} 2c \sum_{k\in\mathbb{Z}^+} \int_{I^3} \int_{\mathbb{R}} e^{is[\mu(c,k,\hbar;\vec{u})-1]/\hbar}\, \check{\phi}(\hbar s)\, ds\, d\vec{u}.$$

Here

$$H(m,n;\vec{u}) = 4\pi^2(u_1 m^2 + u_2 n^2), \quad \mu(c,k,\hbar;\vec{u}) = \hbar^2 \mu(c,k).$$

In (3.2), we assume that ϕ is even, belongs to the Schwartz space $S(\mathbb{R})$ and its Fourier transform $\check{\phi} \in C_0^\infty(\mathbb{R})$. Equation (3.2) follows from the Fourier inversion formula. The density $\mathrm{AV}(\phi)$ is a function of λ, or, equivalently, $\hbar$ and we are interested in estimating it as $\hbar \to 0$. For $\mathrm{AV}_1(\phi)$ the estimation was achieved in [18, Prop. 2.1]. We quote the result:

$$\text{(3.3)} \qquad |\mathrm{AV}_1(\phi;\hbar)| \,=\, \mathcal{O}_\delta(\hbar^{-\delta}).$$

Next, we turn to the estimate for $\mathrm{AV}_2(\phi;\hbar)$ and use the same sort of spectral splitting and integration by parts argument as for $\mathrm{AV}_1(\phi;\hbar)$. We define

$$d\rho_2^0(\phi,\vec{u};\hbar) := \hbar \sum_{c}^{\hbar^{-1-\delta}} \sum_{k}^{\hbar^{-2-\delta}} 2c \int_{\mathbb{R}} e^{is[\mu(c,k,\hbar,\vec{u})-1]/\hbar} \zeta(s) \check{\phi}(\hbar s)\, ds\,,$$

and

$$d\rho_2^+(\phi,\vec{u};\hbar) := \hbar \sum_{c}^{\hbar^{-1-\delta}} \sum_{k}^{\hbar^{-2-\delta}} 2c \int_{\mathbb{R}} e^{is[\mu(c,k,\hbar,\vec{u})-1]/\hbar} (1-\zeta(s)) \check{\phi}(\hbar s)\, ds,$$

where $\zeta(s)$ is equal to 1 close to 0 and is in $C_0^\infty(\mathbb{R})$. We also set

$$d\rho_2^0(\phi;\hbar) = \int_{I^3} d\rho_2^0(\phi,\vec{u};\hbar)\, d\vec{u}, \quad d\rho_2^+(\phi;\hbar) = \int_{I^3} d\rho_2^+(\phi,\vec{u};\hbar)\, d\vec{u}.$$

We split $\mathrm{AV}_2(\phi;\hbar) = d\rho_2^0(\phi;\hbar) + d\rho_2^+(\phi;\hbar)$.

Using a standard stationary phase argument as in [21, Lemma 5.1] we get

$$\text{(3.4)} \qquad d\rho_2^0(\phi;\vec{u}) = \frac{1}{6\pi^2} \mathrm{vol}(M(\vec{u}))\, \check{\phi}(0) \hbar^{-1} + \mathcal{O}(1)$$

uniformly in $\vec{u} \in I^3$.

Integration by parts in u_3 implies the estimate (up to $\mathcal{O}(\hbar^\infty)$ errors)

$$\begin{aligned}(3.5)\quad & d\rho_2^+(\phi;\hbar) \\ & = \hbar \sum_{k=1}^{\hbar^{-2-\delta}} \sum_{c=1}^{\hbar^{-1/2-\delta}} 2\pi c \int_{I^3} \int_{\mathbb{R}} e^{is[\mu(c,k,\hbar)-1]/\hbar} \check{\phi}(\hbar s)(1-\zeta(s))\, ds\, d\vec{u}.\end{aligned}$$

Making the change of variables $V = u_3, U = u_1 u_2$ and integrating by parts in U gives

$$\begin{aligned}(3.6)\qquad d\rho_2^+(\phi;\hbar) &\ll \hbar \sum_{c=1}^{\hbar^{-1/2-\delta}} \sum_{k\geq 1}^{\hbar^{-2-\delta}} \hbar^{-1} \left(\frac{c}{ck}\right) \int_{\mathbb{R}} \frac{1-\zeta(s)}{s}\, ds \\ &\ll \hbar^{-1/2-\delta} |\log \hbar|^2.\end{aligned}$$

Thus, by combining (3.3) and (3.6), we have proved:

Proposition 3.1. *Let $\phi \in S(\mathbb{R})$ with $\check{\phi} \in C_0^\infty(\mathbb{R})$. Then, given the three-dimensional parameter space of Heisenberg manifolds $M(\vec{u})$, we have that for $\hbar$ sufficiently small and any $\delta > 0$,*

$$\mathrm{AV}(\phi;\hbar) = \frac{1}{6\pi^2} \int_{I^3} \mathrm{vol}(M(\vec{u})) d\vec{u} \check{\phi}(0) \hbar^{-1} + \mathcal{O}_\delta(\hbar^{-1/2-\delta}).$$

3.2 Mean-square density of states: upper bounds

The mean square density of states is given by the expression

$$\begin{aligned}(3.7)\quad & \\ \mathrm{MS}(\phi) &= \int_{I^3} \left| d\rho(\phi;\vec{u},\hbar) - \frac{1}{6\pi^2} \mathrm{vol}(M(\vec{u}))\check{\phi}(0)\hbar^{-1} \right|^2 d\vec{u} \\ &= \int_{I^3} |d\rho_1^+(\phi;\vec{u},\hbar) + d\rho_2^+(\phi;\vec{u},\hbar)|^2\, d\vec{u} \;+\; \mathcal{O}(1) \|d\rho_1^+ + d\rho_2^+\|_{L^2}.\end{aligned}$$

So it suffices to prove the estimate $\|d\rho_1^+ + d\rho_2^+\|_{L^2} \ll \hbar^{-1-\delta}$. By introducing $\chi(y)$ a function which is ≥ 1 on $[1-\epsilon, 1+\epsilon]$ and with Fourier transform $\hat{\chi}$ of compact support, we are thus reduced to estimating

$$\begin{aligned}(3.8)\qquad & \int_{I^3} |d\rho_1^+(\phi;\vec{u},\hbar)|^2\, d\vec{u} \\ & \ll \hbar^2 \sum_{m_i,n_i} \int_{\mathbb{R}^5} e^{i\,\Phi(m_1,n_1,m_2,n_2;\vec{s},\hbar)/\hbar}\, a(\vec{s};\hbar)\, \chi(\vec{u})\, d\vec{s} d\vec{u},\end{aligned}$$

and

$$\int_{I^3} |d\rho_2^+(\phi;\vec{u},\hbar)|^2\, d\vec{u} \ll \hbar^2 \sum_{c_i,k_i} c_1c_2 \int_{\mathbb{R}^5} e^{i\,\Psi(c_1,k_1,c_2,k_2;\vec{s},\hbar)/\hbar}\, a(\vec{s};\hbar)\,\chi(\vec{u})\, d\vec{s}d\vec{u}. \tag{3.9}$$

To simplify the writing in (3.8) and (3.9), we have put

$$\Phi(m_1,n_1,m_2,n_2;\vec{s},\hbar) = H(m_1\hbar,n_1\hbar;\vec{u})\, s_1 - H(m_2\hbar,n_2\hbar;\vec{u})\, s_2,$$

$$\Psi(c_1,k_1,c_2,k_2;\vec{s},\hbar) = \mu(c_1,k_1,\hbar;\vec{u})\, s_1 - \mu(c_2,k_2,\hbar;\vec{u})\, s_2,$$

and

$$a(\vec{s};\hbar) := (1-\zeta(s_1))\,(1-\zeta(s_2))\,\check{\phi}(\hbar s_1)\,\check{\phi}(\hbar s_2)\, e^{i(s_2-s_1)/\hbar}.$$

In [18, Prop. 3.2] we showed that for any $\delta > 0$,

$$\int_{I^3} |d\rho_1^+(\phi;\vec{u},\hbar)|^2\, d\vec{u} = \mathcal{O}_\delta(\hbar^{-\delta}). \tag{3.10}$$

Notice that $H(m,n;\vec{u})$ depends only on u_1,u_2, so the integral in the variable u_3 is not essential in estimations. Thus, we are left with estimating the integral

$$J(\phi) = \int_{I^3} |d\rho_2^+(\phi;\vec{u},\hbar)|^2\, d\vec{u}.$$

To do this, we fix $\delta > 0$ and consider the set

$$\Omega(c_1,k_1,c_2,k_2;\hbar) := \{u \in I^3;\ |\mu(c_j,k_j,\hbar;\vec{u}) - 1| \le \hbar^{1-\delta}\,;\ j = 1,2\,\}.$$

By an integration by parts in the s_1, s_2 variables in (3.9), it suffices to assume, modulo $\mathcal{O}(\hbar^\infty)$ errors, that we only sum over quadruples (c_1,k_1,c_2,k_2) with the property that for $\hbar \le \hbar_0$,

$$\Omega(c_1,k_1,c_2,k_2;\hbar) \ne \emptyset.$$

So, we have that $J(\phi) = \int_{I^3} |d\rho_2^+(\phi;\vec{u},\hbar)|^2\, d\vec{u}$ is bounded by

$$C\hbar^2 \sum_{\Omega(c_1,k_1,c_2,k_2;\hbar)\ne\emptyset} c_1c_2 \int_{\mathbb{R}^5} e^{i\Psi(c_1,k_1,c_2,k_2;\vec{s},\hbar)/\hbar} a(\vec{s};\hbar)\chi(\vec{u})\, d\vec{s}d\vec{u} + \mathcal{O}(\hbar^\infty).$$

By making the change of variables $u = u_3, v = u_1 u_2, w = u_1$ in (3.9) and by integrating by parts in u, v, we can assume, modulo $\mathcal{O}(\hbar^\infty)$ errors, that there exists $(s_1, s_2) \in \operatorname{supp}(a)$ such that

$$\frac{\partial}{\partial u}\Psi(c_1, k_1, c_2, k_2; \vec{s}, \hbar) \ll \hbar^{1-\delta}, \quad \frac{\partial}{\partial v}\Psi(c_1, k_1, c_2, k_2; \vec{s}, \hbar) \ll \hbar^{1-\delta}. \tag{3.11}$$

Written out explicitly, the inequality in (3.11) reads

$$\begin{aligned} |c_1\hbar|^2\, s_1 - |c_2\hbar|^2\, s_2 &\ll \hbar^{1-\delta} \\ (2k_1+1)c_1\hbar^2\, s_1 - (2k_2+1)c_2\hbar^2\, s_2 &\ll \hbar^{1-\delta}, \end{aligned} \tag{3.12}$$

Since $\min(|s_1|, |s_2|) \gg 1$ on supp $a(\vec{s};\hbar)$, for a given (c_1, k_1, c_2, k_2) we need to be able to solve (3.12) for some s_1, s_2 with $\min(|s_1|, |s_2|) \gg 1$. By inverting the matrix equation in (3.12) using Cramer's rule and using the estimate $\max(c_j^2, c_j k_j) \ll \hbar^{-2-\delta}$, we get

$$|c_1^2 c_2(2k_2+1) \;-\; c_2^2 c_1(2k_1+1)| \;\ll\; \hbar^{-3-3\delta}. \tag{3.13}$$

On the other hand, the condition $\Omega(c_1, k_1, c_2, k_2; \hbar) \neq \emptyset$ means that for some $u = u(c_1, k_1, c_2, k_2; \hbar)$, $v = v(c_1, k_1, c_2, k_2; \hbar)$,

$$\begin{aligned} (u, v) \cdot (4\pi c_1^2, (2k_1+1)c_1) &= \hbar^{-2} + \mathcal{O}(\hbar^{-1-\delta}) \\ (u, v) \cdot (4\pi c_2^2, (2k_2+1)c_2) &= \hbar^{-2} + \mathcal{O}(\hbar^{-1-\delta}), \end{aligned} \tag{3.14}$$

where $\cdot$ is the standard inner product in $\mathbb{R}^2$. Resubstituting (3.14) back into (3.13) we get

$$|c_1 - c_2| \ll \frac{\hbar^{-1-\delta}}{|c_1| + |c_2|} \quad \text{and} \quad |c_1(2k_1+1) - c_2(2k_2+1)| \ll \hbar^{-1-\delta}. \tag{3.15}$$

Thus

$$\begin{aligned} &J(\phi) \ll \\ &\hbar^2 \sum_{c_2, k_2} \sum_{c_1 \in \beta(c_2, \hbar)} c_1 c_2 \sum_{k_1 \in \gamma(k_2, c_1, c_2, \hbar)} \int_{\mathbb{R}^2} \check{\phi}(\hbar s_1)\check{\phi}(\hbar s_2)(1 - \zeta(s_1)) \\ &\quad \cdot (1 - \zeta(s_2))\hat{\chi}(\hbar[s_1 c_1^2 - s_2 c_2^2], \hbar[s_1(2k_1+1)c_1 - s_2(2k_2+1)c_2])\, d\vec{s}, \end{aligned} \tag{3.16}$$

where

$$\beta(c_2,\hbar) := \left\{c_1;\ |c_1-c_2| \ll \frac{\hbar^{-1-\delta}}{|c_1|+|c_2|}\right\}, \tag{3.17}$$

and

$$\gamma(k_2,c_1,c_2,\hbar) := \left\{k_1;\ |(2k_1+1)c_1-(2k_2+1)c_2| \ll \hbar^{-1-\delta}\right\}. \tag{3.18}$$

We make the change of variables

$$S=\hbar(c_1(2k_1+1)s_1-c_2(2k_2+1)s_2), \quad T=s_1+s_2,$$

We notice that, since $\check{\phi}$ has compact support, $\hbar(s_1+s_2)\ll 1$, which gives $T \ll \hbar^{-1}$. Equation (3.16) implies that $|J(\phi)|$ is bounded by

(3.19)

$$C\hbar^{2-1}\sum_{c_2,k_2}\sum_{c_1\in\beta(c_2,\hbar)}\sum_{k_1\in\gamma(k_2,c_1,c_2,\hbar)}\left(\frac{c_1c_2}{c_1k_1+c_2k_2}\right)\int_{|S|\ll 1}\int_{|T|\ll\hbar^{-1}} dSdT$$
$$\ll \hbar^{-\delta}\sum_{c_2,k_2}\sum_{c_1\in\beta(c_2,\hbar)}\sum_{k_1\in\gamma(k_2,c_1,c_2,\hbar)}\left(\frac{c_1c_2}{c_1k_1+c_2k_2}\right).$$

Ignoring the term c_1k_1 in the denominator of (3.19), one gets

$$\begin{aligned}|J(\phi)| &\ll \hbar^{-\delta}\sum_{c_2,k_2}\sum_{c_1\in\beta(c_2,\hbar)}\sum_{k_1\in\gamma(k_2,c_1,c_2,\hbar)}\frac{c_1}{k_2}\\ &\ll \hbar^{-\delta}\sum_{c_2,k_2}\sum_{c_1\in\beta(c_2,\hbar)}|\gamma(k_2,c_1,c_2,\hbar)|\cdot\frac{c_1}{k_2},\end{aligned} \tag{3.20}$$

where we denote by $|A|$ the cardinality of a set A. From the definitions of the sets $\beta(c_2,\hbar)$ and $\gamma(k_2,c_1,c_2,\hbar)$ in (3.17) and (3.18), it is clear that

$$|\gamma(k_2,c_1,c_2,\hbar)| \ll \frac{\hbar^{-1-\delta}}{c_1},$$

and

$$|\beta(c_2,\hbar)| \ll \frac{\hbar^{-1-\delta}}{c_2}.$$

So, from (3.20) it follows that

$$\begin{aligned}&\int_{I^3}|d\rho_2^+(\phi;\vec{u},\hbar)|^2\,d\vec{u}\\ &\ll \hbar^{-\delta}\sum_{c_2=1}^{\hbar^{-1-\delta}}\sum_{k_2=1}^{\hbar^{-2-\delta}}\left(\frac{c_1}{k_2}\right)\cdot\left(\frac{\hbar^{-1-\delta}}{c_1}\right)\cdot\left(\frac{\hbar^{-1-\delta}}{c_2}\right)\\ &\ll \hbar^{-2-\delta}|\log\hbar|^2=\mathcal{O}_\delta(\hbar^{-2-\delta}).\end{aligned} \tag{3.21}$$

Finally, by the Cauchy-Schwartz inequality and the estimates in (3.10) and (3.21),

$$\int_{I^3} d\rho_2^+(\phi;\vec{u},\hbar)\cdot\overline{d\rho_1^+(\phi;\vec{u},\hbar)}\,d\vec{u}=\mathcal{O}_\delta(\hbar^{-1-\delta}).$$

Consequently, we have proved:

Proposition 3.2. *Let $M(\vec{u})$ be in the three-dimensional parameter space of Heisenberg manifolds described in (1.3). Then, for any $\delta>0$,*

$$MS(\phi):=\int_{I^3}|d\rho(\phi;\vec{u},\hbar)-\frac{1}{6\pi^2}\mathrm{vol}(M(\vec{u}))\check{\phi}(0)\hbar^{-1}|^2\,d\vec{u}=\mathcal{O}_\delta(\hbar^{-2-\delta}).$$

3.3 The spectral decomposition: Proof of Theorem 1.3

Let $\phi\in S(\mathbb{R})$ with $\phi(\lambda)>0$, and $\check{\phi}\in C_0^\infty(\mathbb{R})$ with $\check{\phi}(0)=1$. We start with a rescaled spectral decomposition used by Duistermaat and Guillemin [8], but applied to the eigenvalues of Δ rather than $\sqrt{\Delta}$. Taking into account this rescaling we will naturally encounter semiclassical density of states on scales of order $\sim\hbar^2$ where $\hbar^{-1}=\sqrt{\lambda}$. Our starting point is the following basic decomposition (see [8], [21]):

$$\begin{aligned}(3.22)\qquad &\int_{-\infty}^{\infty}\int_{-\infty}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u})\\
&=\int_{\lambda'\geq\lambda+1}\int_{-\infty}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u})\\
&\quad+\int_{|\lambda-\lambda'|<1}\int_{-\infty}^{\lambda}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u})\\
&\quad+\int_{\lambda'<\lambda-1}\int_{-\infty}^{\infty}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u})\\
&\quad-\int_{\lambda'\leq\lambda-1}\int_{\lambda}^{\infty}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u}).\end{aligned}$$

Since $1=\int_{-\infty}^{\infty}\phi(s)ds$, it follows that the third term on the right-hand side of (3.22)

$$\int_{\lambda'<\lambda-1}\int_{-\infty}^{\infty}\phi(x-\lambda')\,dx\,dN(\lambda';\vec{u})=N(\lambda-1;\vec{u}).$$

To estimate the other terms in (3.22), we will need the following proposition:

Proposition 3.3. *Let $\lambda_j(\vec{u}) \in \operatorname{Spec}(M(\vec{u}))$. Then, for any $\phi \in S(\mathbb{R})$ as above, we have that*

$$\int_{I^3} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\lambda} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) - \frac{1}{6\pi^2}\, \lambda^{3/2} \operatorname{vol}(M(\vec{u})) \right|^2 d\vec{u}$$
$$= \mathcal{O}_\delta(\lambda^{3/2+\delta}).$$

Proof. The proof is similar to [18, Prop. 4.1]. We only show the necessary changes here.

Put $\hbar^{-1} = \sqrt{\lambda}$ and consider the rescaled operator $H := \hbar^2 \Delta$ with eigenvalues $\lambda_j(\hbar) = \hbar^2 \lambda_j : j = 1, 2, \ldots.$ As in [18, Eq. (4.4)], we see that, modulo $\mathcal{O}(1)$ errors, we are reduced to estimating

$$I(\vec{u},\hbar) \;=\; \hbar^{-2} \sum_{j=1}^{\infty} \int_{-1}^{1} \phi\left(\frac{\lambda_j(\hbar;\vec{u}) - \Lambda}{\hbar^2} \right) d\Lambda. \tag{3.23}$$

Next, we split integral in (3.23) into two pieces corresponding to the zero and nontrivial period spectrum: Let $\zeta \in C_0^\infty(\mathbb{R})$ be nonnegative, equal to 1 close to 0. Its support should be small enough. Then,

$$I(\vec{u},\hbar) \;=\; I_1^0(\vec{u},\hbar) \;+\; I_1^+(\vec{u},\hbar) + I_2^0(\vec{u},\hbar) \;+\; I_2^+(\vec{u},\hbar), \tag{3.24}$$

where

$$I_1^0(\vec{u},\hbar) = \hbar^{-1} \sum_{m,n\in\mathbb{Z}} \int_{-1}^{1} \int_{-\infty}^{\infty} e^{is[H(m\hbar,n\hbar;\vec{u})-\Lambda]/\hbar} \zeta(s) \check{\phi}(\hbar s)\, ds\, d\Lambda,$$

and

$$I_2^0(\vec{u},\hbar) = \hbar^{-1} \sum_{c,k\geq 0} \int_{-1}^{1} \int_{-\infty}^{\infty} e^{is[\mu(c,k,\hbar;\vec{u})-\Lambda]/\hbar} \zeta(s) \check{\phi}(\hbar s)\, ds\, d\Lambda,$$

and I_1^+ and I_2^+ are defined similarily by replacing the cutoff $\zeta(s)$ above by $1 - \zeta(s)$. First, by a fairly standard stationary phase argument, see [21, Lemma 5.1]

$$I^0(\vec{u},\hbar) = \frac{1}{6\pi^2} \operatorname{vol}(M(\vec{u}))\, \hbar^{-3} + \mathcal{O}(\hbar^{-1}). \tag{3.25}$$

To prove Proposition 3.3, we need to estimate

$$\int_{I^3} |I^+(\vec{u},\hbar)|^2\, d\vec{u} = \int_{I^3} \left| I(\vec{u},\hbar) - I^0(\vec{u},\hbar) \right|^2 d\vec{u}$$
$$\ll \int_{I^3} |I_1^+(\vec{u},\hbar)|^2\, d\vec{u} + \int_{I^3} |I_2^+(\vec{u},\hbar)|^2\, d\vec{u}.$$

We get

$$\begin{aligned}(3.26)\quad &\int_{I^3} |I^+(\vec{u},\hbar)|^2 d\vec{u} \\ &\ll \sum_{m_i,n_i\neq 0}^{C\hbar^{-1-\delta}} \int e^{i\Phi(m_1,n_1,m_2,n_2;\vec{s},\hbar)/\hbar} b(\vec{s};\hbar)\frac{1}{s_1 s_2} d\vec{s}d\vec{u} \\ &\quad + \sum_{c_i,k_i>0} c_1c_2 \int e^{i\Psi(c_1,k_1,c_2,k_2;\vec{s},\hbar)/\hbar}\, b(\vec{s};\hbar)\frac{1}{s_1s_2} d\vec{s}d\vec{u} + \mathcal{O}(\hbar^\infty),\end{aligned}$$

with

$$\begin{aligned} b(\vec{s};\hbar) = (&e^{i(-s_1+s_2)/\hbar} + e^{i(s_1-s_2)/\hbar} - e^{i(-s_1-s_2)/\hbar} - e^{i(s_1+s_2)/\hbar}) \\ &\cdot (1-\zeta(s_1))(1-\zeta(s_2))\check{\phi}(\hbar s_1)\check{\phi}(\hbar s_2)\chi(u_1)\chi(u_2).\end{aligned}$$

Since

$$s_1 s_2 \geq \frac{1}{2}(s_1+s_2) \quad \text{when} \quad \min(s_1,s_2)\geq 1,$$

it follows by the argument in Proposition 3.2 that the right-hand side in (3.26) is

$$\begin{aligned}(3.27)\quad &\ll \hbar^{-2-2\delta} \sum_{m_2,n_2} \frac{1}{m_2 n_2} \min\left(\frac{1}{\hbar m_2^2}, \frac{1}{\hbar n_2^2}\right) \int_{C<|T|\ll \hbar^{-1}} \frac{dT}{T} \\ &+ \hbar^{-1} \sum_{c_2} \sum_{c_1\in\beta(c_2,\hbar)} \sum_{k_2} \sum_{k_1\in\gamma(k_2,c_1,c_2,\hbar)} \left(\frac{c_1c_2}{c_1k_1+c_2k_2}\right) \int_{C<|T|\ll\hbar^{-1}} \frac{dT}{T} \\ &\ll \hbar^{-1-\delta} + \hbar^{-3-\delta}.\end{aligned}$$

The sets $\beta(m_2,\hbar)$ and $\gamma(n_2,\hbar)$ are defined in (3.17), (3.18) and the argument follows as in (3.19) with the only change being the appearance of the denominator $|s_1|+|s_2|$. This completes the proof of Proposition 3.3.

q.e.d.

By reshuffling the terms in the spectral decomposition and integrating over the parameters $\vec{u}\in I^3$, we get:

$$
\begin{aligned}
(3.28)\qquad & \int_{I^3} |N(\lambda-1;\vec{u}) - I^0(\vec{u},\hbar)|^2 \, d\vec{u} \\
&\ll \int_{I^2} |I_1^+(\vec{u},\hbar) + I_2^+(\vec{u},\hbar)|^2 d\vec{u} \\
&\quad + \int_{I^3} \left| \int_{\lambda' \geq \lambda+1} \int_{-\infty}^{\lambda} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right|^2 d\vec{u} \\
&\quad + \int_{I^3} \left| \int_{|\lambda-\lambda'|<1} \int_{-\infty}^{\lambda} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right|^2 d\vec{u} \\
&\quad + \int_{I^3} \left| \int_{\lambda' \leq \lambda-1} \int_{\lambda}^{\infty} \phi(x-\lambda')\, dx\, dN(\lambda';\vec{u}) \right|^2 d\vec{u} \\
&= T_0 + T_1 + T_2 + T_3.
\end{aligned}
$$

From Proposition 3.3 we have that for any $\delta > 0$,

$$
T_0 = \int_{I^3} |I_1^+(\vec{u},\hbar) + I_2^+(\vec{u},\hbar)|^2 \, d\vec{u} \,=\, \mathcal{O}_\delta(\hbar^{-3-\delta}).
$$

Consequently, it remains to estimate the terms T_1, T_2, T_3 on the right-hand side of the inequality (3.28). This is accomplished by exactly the same argument as in [18, (4.12) and (4.15)]. The end result is that each of these three terms can be estimated by the mean-square density of states in Proposition 3.2: that is for $j = 1, 2, 3$, and any $\delta > 0$,

$$
T_j = \mathcal{O}_\delta(\hbar^{-2-\delta}).
$$

This completes the proof of Theorem 1.3. q.e.d.

4. Geometry of the geodesic flow

Consider the standard Heisenberg group, H_1, and its quotient $\Gamma_1 \setminus H_1$. The Lie algebra $\mathcal{H}_1 := T_e H_1$ has a basis given by the left-invariant differential one-forms:

$$
(4.1)\qquad \alpha = dx,\ \beta = dy,\ \gamma = dz \,-\, x\, dy.
$$

Let ω be the canonical one-form on T^*H_1 and define the fiber momentum coordinates corresponding to the basic vector fields $\partial_x, \partial_y, \partial_z$ in the usual way: $p_x := \omega(\partial_x), p_y := \omega(\partial_y), p_z := \omega(\partial_z)$. It is straightforward to check that

$$p_x = p_\alpha, \;\; p_z = p_\gamma, \;\; p_y = p_\beta - x\, p_\gamma. \tag{4.2}$$

The Hamiltonian for the geodesic flow is

$$\widetilde{H} = \frac{1}{2}(p_\alpha^2 + p_\beta^2 + p_\gamma^2) \tag{4.3}$$

and it is Liouville integrable (see [2], [20]). Indeed, one can use as integrals in involution the functions

$$F = p_\gamma \;\; \text{and} \;\; G = \phi(p_\gamma)\, \sin 2\pi \left(\frac{p_\beta}{p_\gamma} - x \right).$$

In the definition of G, the function $\phi \in C^\infty(\mathbb{R})$ is required to vanish to infinite order at 0 to kill the singularity at $p_\gamma = 0$ in the sine function. So, one can take for example

$$\phi(u) = \exp(-u^{-2}).$$

It is easily checked that the functions F, G, H are in involution and have differential which are independent on a dense subset of T^*M. Notice that the function G is C^∞ but *not* real analytic. Even though the geodesic flow on M is completely integrable, on the regular set of the moment map, a routine computation (see for example [2]) shows the geodesic flow restricts to two-dimensional isotropic sub-tori of the standard three-dimensional Lagrangian level sets. This fact suggests that there may be large (> 3) dimensional periodic sets in T^*M. As we will now show, this is indeed the case.

The Hamilton equations for the geodesic flow in terms of the $(x, y, z; p_\alpha, p_\beta, p_\gamma)$ coordinates on T^*H_1 are:

$$\begin{aligned} &\frac{dx}{dt} = p_\alpha, \;\; \frac{dy}{dt} = p_\beta, \;\; \frac{dz}{dt} = p_\gamma + x p_\beta, \\ &\frac{dp_\alpha}{dt} = -p_\gamma\, p_\beta, \;\; \frac{dp_\beta}{dt} = p_\alpha\, p_\gamma, \;\; \frac{dp_\gamma}{dt} = 0. \end{aligned} \tag{4.4}$$

and we are interested in the solution curves to (4.4) on the cosphere bundle $\widetilde{H} = 1$. From the last equation in (4.4) we get that

$$p_\gamma(t) = p_z(t) = c_1 \tag{4.5}$$

for some constant c_1. Resubstituting (4.5) back into (4.4), we can easily solve the resulting equations $\dot{p_\alpha}$ and $\dot{p_\beta}$ and get that

$$p_\alpha(t) = c_2 \cos(tc_1) + c_3 \sin(tc_1)$$
$$p_\beta(t) = c_2 \sin(tc_1) - c_3 \cos(tc_1). \tag{4.6}$$

Here c_2, c_3 are additional constants satisfying $c_2^2 + c_3^2 = 2 - c_1^2$ with $p_\alpha(0) = c_2$ and $p_\beta(0) = -c_3$.

Since $\dot{x} = p_\alpha$ and $\dot{y} = p_\beta$ it then follows from (4.6) that

$$x(t) = \frac{c_2}{c_1}\sin(tc_1) - \frac{c_3}{c_1}\cos(tc_1) + \frac{c_3}{c_1} + x(0),$$
$$y(t) = -\frac{c_2}{c_1}\cos(tc_1) - \frac{c_3}{c_1}\sin(tc_1) + \frac{c_2}{c_1} + y(0). \tag{4.7}$$

Finally, resubstituting (4.7) and (4.6) back into the equation for $\dot{z}$ gives:

$$z(t) = \left(c_1 + \frac{c_2^2 + c_3^2}{2c_1}\right) t + \frac{c_3^2 - c_2^2}{4c_1^2}\sin(2c_1 t) - \frac{c_2 c_3}{c_1^2}\sin^2(c_1 t)$$
$$+ \left(\frac{c_3}{c_1} + x(0)\right)\left(-\frac{c_2}{c_1}\cos(c_1 t) - \frac{c_3}{c_1}\sin(c_1 t)\right) + m \tag{4.8}$$

where $z(0) = -(c_3/c_1 + x(0))(c_2/c_1) + m$. We would like to determine what conditions to impose on c_1 to ensure that the geodesics

$$\gamma(t) := (x(t), y(t), z(t), p_\alpha(t), p_\beta(t), p_\gamma(t))$$

on the level set $\{p_z = c_1\} \cap S^*M$ are *all* periodic. Since the solutions to the equations (4.7) and (4.6) are all periodic of period $(2\pi)/c_1$, to ensure that $z(t)$ closes up on the quotient manifold M we require that for all t,

$$z\left(t + \frac{2\pi}{c_1}\right) - z(t) \in \mathbb{Z}. \tag{4.9}$$

Since $c_2^2 + c_3^2 = 2 - c_1^2$ this implies that

$$\frac{1}{c_1^2} + \frac{1}{2} \in \frac{\mathbb{Z}}{2\pi}. \tag{4.10}$$

We henceforth denote the discrete set of c_1's satisfying (4.10) by $\mathcal{T}$.

Proposition 4.1. *Given c_1 satisfying (4.10), the 4-dimensional manifold*

$$P(c_1) := \{(x, y, z, p_\alpha, p_\beta, p_z), p_z = c_1\} \cap S^*M$$

consists of periodic geodesics of primitive period $(2\pi)/c_1$. Moreover, this set contains all the primitive periods of closed geodesics on the Heisenberg manifold, $M = \Gamma_1 \backslash H_1$, with metric given by the Hamiltonian (4.3).

Proof. The argument above shows that when c_1 satisfies (4.10) all the geodesics on $P(c_1)$ are periodic of primitive period $(2\pi)/c_1$. To show that these numbers include *all* possible primitive periods, we argue as follows: By matrix multiplication, two points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in H_1$ are identified under Γ if and only if there exist $k_1, k_2, k_3 \in \mathbb{Z}$ such that:

$$\begin{aligned} x_1 &= x_2 + k_1 \\ y_1 &= y_2 + k_2 \\ z_1 &= z_2 + k_3 + k_1 y_2. \end{aligned} \tag{4.11}$$

Fix c_1, c_2, c_3. Then, the solution curves $p_\alpha(t)$ and $p_\beta(t)$ in (4.6) are just circles of period $(2\pi)/c_1$ and so, these numbers include all possible periods. The coordinates $x(t)$ and $y(t)$ are automatically periodic with primitive period $T = (2\pi)/c_1$. So, to determine the conditions on c_1 that ensure periodicity, we only need to consider the last congruence for $z(t)$ in (4.11). Since $\cos t$ and $\sin t$ are continuous in $t \in \mathbb{R}$ it necessarily follows that $k_1 = 0$ and so condition (4.10) is necessary as well as sufficient to guarantee periodicity. q.e.d.

5. Spectral averaging: Proof of Theorem 1.4

We follow a simple trace-formula argument of Sarnak [19, Lemma 5.1], but localized near a period $T_0 = (2\pi)/c_1$ where c_1 satisfies (4.10). More precisely, let $\phi \in S(\mathbb{R})$ with $\check{\phi} \in C_0^\infty(\mathbb{R})$, $\check{\phi}(T_0) = 1$ with

$$\text{supp } \check{\phi} \cap \mathcal{T} = T_0.$$

We set $N_1(s) = N(s^2)$ and $R_1(s) = R(s^2)$, $X = \sqrt{\lambda}$. It is easy to check that $P(c_1)$ is clean [8] and since $\dim P(c_1) = 4$, it then follows from the wave-trace formula [8], [13] that:

$$\int_{-\infty}^{\infty} \phi(X - s)\, dN_1(s) \sim_{X \to \infty} \sum_{j=0}^{\infty} c_j\, X^{3/2 - j}. \tag{5.1}$$

Moreover, since the support of $\check{\phi}$ is away from $s = 0$,

$$\int_{-\infty}^{\infty} \phi(X-s)\, dN_1(s) = \int_{-\infty}^{\infty} \phi(X-s)\, dR_1(s) \tag{5.2}$$
$$= -\int_{-\infty}^{\infty} \phi'(X-s)\, R_1(s)\, ds.$$

Finally, using the fact that ϕ' is a Schwartz function and $R_1(s) = \mathcal{O}(s^2)$ by the Hörmander bound, we get from (5.1) and (5.2) that:

$$\frac{1}{X}\int_{X}^{2X} |R_1(s)|\, ds \gg X^{3/2}. \tag{5.3}$$

The change of variables $\tau = s^2$ suffices to get the result of Theorem 1.4.

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